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An approximate analytic solution of the Lorenz equations

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Abstract. Multiple-time perturbation theory is applied to the Lorenz equations to obtain approximate solutions. These solutions are used to obtain the form of the mapping relating two successive maxima of one of the variables of the equations. Good agreement is obtained with the numerical results presented by Lorenz.

1. Introduction

Some time ago Lorenz (1963) gave the results of a detailed numerical solution of the set of deterministic equations

$$\begin{aligned} dx/dt &= \sigma(y-x), \\ dy/dt &= -xz + rx - y, \\ dz/dt &= xy - bz \end{aligned} \tag{1.1}$$

where σ, r and b are constants. These equations originally arose in a study of thermal convection near criticality. The quantities x, y and z are proportional to the amplitude of the three most dominant normal modes of the system. Near criticality, the parameter r, the Rayleigh number, should be of order unity. However Lorenz solved the equations for $r = 28$ and found, surprisingly, that this set of deterministic equations gave rise to complicated non-periodic behaviour whose form depended critically on initial conditions. More recently these equations have received much attention as a relatively simple example of a 'strange attractor'. Ruelle and Takens (1971) have conjectured that such attractors form the basis for any theory of turbulence. Haken (1975) has shown that the Lorenz equations, with a different interpretation of the symbols, arise naturally in the theory of high-power lasers. For additional references see the recent review article by Shaw (1981).

The basic property of this example of a strange attractor is that for $r > r_c$ ($r_c = \sigma(\sigma+b+3)/(\sigma-b-1)$) there exist two unstable critical points about each of which the solution spirals out until its amplitude is sufficiently large when it is then attracted to the other critical point about which it then spirals. The jumping from the vicinity of one critical point to that of the other seems to take place in a random fashion. It is this process which leads to the non-periodic behaviour of the solution and its sensitivity to initial conditions. From an analysis of his numerical data Lorenz evaluated the maximum value of z during each spiral (measured from the appropriate critical point) and found to within his numerical accuracy that he could write

$$M_{n+1} = F(M_n) \tag{1.2}$$

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where M_n is the maximum value of z after n spirals. Lorenz approximated the function F obtained numerically by an even simpler function which could be written in the form $F = 2M$ for $M \leqslant \frac{1}{2}$ and $F = 2(1-M)$ for $M \geqslant \frac{1}{2}$. Using this model function Lorenz went on to show that (1.2) had a non-denumerable set of solutions which were non-periodic and that these solutions were sensitive to initial conditions. Thus the basic complexity of the solutions of the differential equations (1.1) arises from the complexity of the solutions of (1.2).

Equations of the form (1.2) also arise in a study of population dynamics of certain biological systems, in which case M_n is to be interpreted as the population number after n generations. A class of such equations has received considerable attention, and the following general picture of the type of solution has now emerged. Consider equations of the form (1.2) with $F(M)$ having a single maximum in the range of M of interest, and F also a function of a parameter α. For small α one finds that a single critical point exists such that $F(M) = M$ and that this point is a stable fixed point. For larger values of α this fixed point bifurcates into two stable fixed points, whilst further increases in α give rise to further bifurcations. This process continues until α reaches a critical value, α_c say, above which the solution becomes chaotic. That is, there exists a non-denumerable set of non-periodic solutions as well as an infinite number of periodic ones.

The form of $F(M)$ found numerically by Lorenz is not of the above form: the simple maximum is replaced by a cusp. However for the particular analytic form for $F(M)$ considered by Lorenz the bifurcations are all degenerate and the solution exists in the chaotic regime. Thus though we have no reason to expect functions with cusps to give behaviour quantitatively the same as those with simple maxima, we do expect qualitative features such as chaos to exist. The general features of equations such as (1.2) are also considered in the review article by Shaw (1981).

In conclusion, we see that features associated with the spiralling orbits of the strange attractors have a direct correspondence with the solutions of a difference equation of the form (1.2). The non-periodic motion characteristic of the strange attractor is closely linked with the chaotic motion of the difference equation. Thus though it is undoubtedly an approximation to replace (1.1) by an equation of the form (1.2), such an approximation retains the physically important concept of non-periodic solutions.

The purpose of this paper is to present a method of approximation which, when applied to (1.1), yields an equation of the form (1.2) without recourse to any numerical analysis. An explicit form for $F(M)$ is obtained which is in surprisingly good agreement with that obtained numerically by Lorenz.

In a recent paper Yorke and Yorke (1979) have obtained an analytic expression for the function $F(M_n)$. However their work differs in two distinct ways from the work presented in this paper. Firstly, they obtain the form for $F(M_n)$ by fitting data obtained numerically to assumed functional forms, and not by analytical methods. Secondly they consider the regime $r < r_c$ where the basic topology of the solution is fundamentally different from the one considered here.

2. Method of approximation

The critical points of (1.1), that is points where the time derivatives are zero, are given by $x_0^2 = b(r-1)$, $y_0 = x_0$ and $z_0 = r-1$. (There is a further critical point, namely

$x_0 = y_0 = z_0 = 0$, but this plays no role in the following analysis.) For $r > r_c$ these critical points are unstable and a linear analysis about either point shows that the time dependence of x, y or z is proportional to $\exp[(i\lambda_0 + \alpha)t]$. To lowest significant order in an expansion in $r - r_c$

$$\lambda_0^2 = C_0^2 + b(1+\sigma), \tag{2.1}$$

$$\alpha = [C_0 \Delta C(\sigma - b - 1)]/(\lambda_0^2 + \lambda_1^2), \tag{2.2}$$

$\lambda_1 = 1 + b + \sigma$, $C_0^2 = b(r_c - 1)$, $\Delta C = C - C_0$ with $C^2 = b(r-1)$. For r sufficiently close to r_c, the periodic motion of the orbit associated with a timescale $1/\lambda_0$ is much faster than the change of amplitude of the orbit associated with the timescale $1/\alpha$. With this in mind a many-timescale method of perturbation theory has been used to solve (1.1) (see Nayfeh 1973). To this end one formally writes

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

where x_0, x_1, x_2, etc, are functions of the many times t, $t_1 = \varepsilon t$, $t_2 = \varepsilon^2 t$, etc. It is also necessary to order ΔC to be of order ε^2. Then the standard theory gives, to order ε^3,

$$x = x_0 + A\cos\theta + A^2(B\cos 2\theta + D\sin 2\theta + E), \tag{2.3}$$

$$y = y_0 + [A\cos\theta - (\lambda_0/\sigma)\sin\theta] + A^2(F\cos 2\theta + G\sin 2\theta + E), \tag{2.4}$$

$$z = z_0 + (A\lambda_0/\sigma C_0)[\lambda_0\cos\theta + (1+\sigma)\sin\theta] + A^2(H\cos 2\theta + I\sin 2\theta + J) \tag{2.5}$$

where $\theta = \lambda_0 t + \chi$. Expressions for B, D, E, F, G, H, I and J are given in the appendix.

In obtaining the above solution it is necessary to apply a consistency condition on the timescale t_2. This gives rise to the two following equations for the slow time variation of A and χ:

$$2\lambda_0^2 \frac{dA}{dt} + 2\lambda_0\lambda_1 A \frac{d\chi}{dt} - 4\sigma C_0 \Delta C A - \sigma A^3 T_1 = 0 \tag{2.6}$$

and

$$2\lambda_0\lambda_1 \frac{dA}{dt} - 2\lambda_0^2 A \frac{d\chi}{dt} + 2\lambda_0 C_0 \Delta C A - \sigma A^3 T_2 = 0 \tag{2.7}$$

where expressions for T_1 and T_2 are given in the appendix.

These two equations may be combined to give the following equation for the time evolution of A:

$$dA/dt = \alpha A + \beta A^3 \tag{2.8}$$

where α is the linear growth rate given by (2.2) and

$$\beta = \sigma(\lambda_0 T_1 + \lambda_1 T_2)/2\lambda_0(\lambda_0^2 + \lambda_1^2). \tag{2.9}$$

McLaughlin and Martin (1975) have also given a perturbation theory based on the smallness of the ratio α/λ_0. However their treatment of the non-linear terms is not consistent: they include some but ignore others. The advantage of the many-time perturbation theory is that all terms of the same order are treated equally. For example their equation (4.29), which has essentially the same form as (2.8), gives a value $\beta = 0.0257$ for the Lorenz parameters $\sigma = 10$, $r = 28$, $b = \frac{8}{3}$ whilst the present theory gives $\beta = 0.0039$, at least a factor of 10 different. Both equations show that the Lorenz model exhibits an inverted bifurcation. This was one of the major qualitative features these authors discussed.

The above equations give an approximate analytic solution of the Lorenz equation which is expected to be a good approximation for r values in the vicinity of but greater than r_c. To compare this analytic solution with the numerical results of Lorenz, in the next section we discuss a derivation of a mapping of the form of (1.2).

3. The discrete mapping

The time dependence of the z coordinate is given by (2.5). To find the maximum value of z during one orbit we treat A and χ as constants and maximise with respect to θ. Further, since this is an expression for z to $O(A^3)$, we can only find the value of the maximum value of z, M say, to this order. This procedure gives

$$M - z_0 \equiv \bar{M} = p_0 A + p_1 A^2 + O(A^3) \tag{3.1}$$

where

$$p_0 = \lambda_0[\lambda_0^2 + (1+\sigma)^2]^{1/2}/\sigma C_0 \tag{3.2}$$

and

$$p_1 = \{[\lambda_0^2 - (1+\sigma)^2]H + 2\lambda_0(1+\sigma)I + [\lambda_0^2 + (1+\sigma)^2]J\}/[\lambda_0^2 + (1+\sigma)^2]. \tag{3.3}$$

The time dependence of M is now through that of A, which from (2.8) takes the form

$$A^2(t) = \alpha\, e^{2\alpha t}/(a^2 - \beta\, e^{2\alpha t}) \tag{3.4}$$

where a^2 is an integration constant. However before we can calculate M after n orbits we need to know the period of the orbit. From the form of z given by (2.5) we see that the time dependence is governed by θ. To calculate θ we need to know the time variation of χ. This is obtained from equations (2.6), (2.7) and (3.4), and we write it in the form

$$\theta = \theta_0 + (\lambda_0 + \gamma)t + \frac{\sigma(\lambda_1 T_1 - \lambda_0 T_2)}{4\lambda_0(\lambda_0^2 + \lambda_1^2)} \ln\left(1 + \frac{\beta A^2}{\alpha}\right) \tag{3.5}$$

where

$$\gamma = C_0 \Delta C \frac{(\lambda_0^2 + 2\lambda_1\sigma)}{\lambda_0(\lambda_0^2 + \lambda_1^2)}. \tag{3.6}$$

This last expression is the correction to λ_0 to obtain agreement with a linear analysis.

The last term in equation (3.5) shows that the orbital period is amplitude dependent and in particular shows singular behaviour for large A. Such behaviour has been found numerically both by Lorenz (1963; see table 2) and by Shaw (1981; see figure 31). However for the Lorenz parameters the period is almost constant over most of the range of allowed values of A. In the following we ignore this term and consider the orbital motion to be periodic with period $\tau = 2\pi/(\lambda_0 + \gamma)$. Then using (3.4) we can express the value of A after n orbits, A_n, in terms of the value after $(n+1)$ orbits, A_{n+1}. We find

$$A_{n+1}^2 = A_n^2/[(1 + \beta A_n^2/\alpha)\, e^{-2\alpha\tau} - \beta/\alpha]. \tag{3.7}$$

This relation, in conjunction with (3.1), can be used to obtain $\bar{M}_{n+1}$ as a function of $\bar{M}_n$. This simplifies for the Lorenz parameters, where $p_1/p_0 \approx -0.01$, and if such terms are ignored one finds

$$\bar{M}_{n+1} = \bar{M}_n/[e^{-2\alpha\tau}(1+\beta\bar{M}_n^2/\alpha p_0^2) - \beta/\alpha p_0^2]^{1/2}, \tag{3.8}$$

which is in the form of equation (1.2). For the Lorenz parameters we have

$$\bar{M}_{n+1} = 1.064\bar{M}_n/(1-0.001\,72\bar{M}_n^2)^{1/2}. \tag{3.9}$$

This formula has been tested against the numerical values of Lorenz. From his table 2 and for values of his N from 0107 to 1643 (the orbit then jumps to the other critical point) values of $\bar{M}_{n+1}/\bar{M}_n$ agree with the analytic formula (3.9) to within a few per cent. This corresponds to values of $\bar{M}_n$ ranging from unity up to 13, that is all values up to the singular point in the Lorenz map. The value of the period over most of this range is, from table 2, equal to 0.62, whereas the approximation used above, namely $\tau = 2\pi/(\lambda_0+\gamma)$, has the value 0.65 for the Lorenz parameters. Again this is good agreement.

We conclude that the method of approximation given in this paper gives a good representation of the Lorenz map, at least for $\bar{M}$ values less than the critical value at the cusp, $\bar{M}_c$. Because of the symmetry of the Lorenz equations, the map for values of $\bar{M} > \bar{M}_c$ is simply the mirror image of the map for $\bar{M} < \bar{M}_c$. The two branches of the map correspond to orbits about the two critical points.

It remains to obtain a value for $\bar{M}_c$, which we estimate from figure 4 of Lorenz to be of order 11.5. The condition we apply is that the orbit changes from being about one critical point to the other when the x coordinate goes through zero. From (2.3) we see, to $O(A^3)$, that the minimum value of x occurs for $\theta = \pi$ so that the maximum value of A satisfies $x_0 = A - A^2(B+E)$. If this equation is iterated and the value for A so obtained substituted in (3.1) we obtain the following estimate of $\bar{M}_c$:

$$\bar{M}_c = p_0 x_0 + [p_0(B+E)+p_1]x_0^2. \tag{3.10}$$

For the Lorenz parameters this gives $\bar{M}_c = 11.8$, which is in excellent agreement with the computed value.

4. Conclusions

A method of analytic approximation has been applied to the Lorenz equations and detailed orbit solutions have been obtained; these are given by (2.3), (2.4) and (2.6) in conjunction with (3.4) and (3.5). These have been used to construct a discrete mapping for successive values of the maximum of the z variable, $\bar{M}$. The mapping is of the general form of (1.2), with $F(\bar{M})$ defined by (3.8) for $\bar{M} < \bar{M}_c$ and the mirror image about $\bar{M} = \bar{M}_c$ for $\bar{M} > \bar{M}_c$. The critical value of $\bar{M}_c$ is given by (3.10). For the Lorenz parameters the form obtained for $F(\bar{M})$ agrees within a few per cent with the values obtained numerically. Good agreement is also obtained for the period of oscillation and the critical value $\bar{M}_c$.

Appendix

The various constants appearing in equations (2.3)–(2.5) are

$$B = [5(1+\sigma)^2 + 3b^2 + 12\lambda_0^2]/[12C_0(\lambda_1^2 + 4\lambda_0^2)],$$

$$D = (1+\sigma)(\lambda_0^2 + b\lambda_1)/[(3\lambda_0 C_0)(\lambda_1^2 + 4\lambda_0^2)],$$

$$E = -(b\lambda_0^2 + \sigma C_0^2)/2C_0\lambda_0^2\lambda_1,$$

$$F = B + 2D\lambda_0/\sigma,$$

$$G = D - 2B\lambda_0/\sigma,$$

$$H = -[(1+\sigma)2\lambda_0 D - 4\lambda_0^2 B + \lambda_0^2/2C_0]/C_0\sigma,$$

$$I = [2\lambda_0(1+\sigma)B + 4\lambda_0^2 D - \lambda_0(1+\sigma)/2C_0]/C_0\sigma,$$

and

$$J = -\lambda_0^2/2\sigma C_0^2.$$

In addition,

$$T_1 = -\tfrac{1}{2}\lambda_0 I - \lambda_0^2 B(1+\sigma)/\sigma C_0 + \tfrac{3}{4} + 6C_0 E + 3BC_0 + D\lambda_0(C_0^2 + \lambda_0^2)/\sigma C_0$$

and

$$T_2 = \tfrac{1}{2}\lambda_0 H - \lambda_0/4\sigma - 2C_0 E\lambda_0/\sigma + 3C_0 D$$
$$- \lambda_0^2 D(1+\sigma)/\sigma C_0 - \lambda_0 J - B\lambda_0(\lambda_0^2 + C_0^2)/\sigma C_0.$$

References

Haken H 1975 *Phys. Lett.* **53A** 77
Lorenz E N 1963 *J. Atmospheric Sci.* **20** 130
McLaughlin J B and Martin P C 1975 *Phys. Rev.* A **12** 186
Nayfeh A 1973 *Perturbation Methods* (New York: Wiley) ch 6
Ruelle D and Takens F 1971 *Commun. Math. Phys.* **20** 167
Shaw R 1981 *Z. Naturf.* **36** 80
Yorke J A and Yorke E D 1979 *J. Stat. Phys.* **21** 263